ESTIMATING THE INTENSITY IN THE FORM OF A POWER FUNCTION OF AN INHOMOGENEOUS POISSON PROCESS

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Abstract. An estimator of the intensity in the form of a power function of an inhomogeneous Poisson process is constructed and investigated. It is assumed that only a single realization of the Poisson process is observed in a bounded window. We prove that the proposed estimator is consistent when the size of the window indefinitely expands. The asymptotic bias, variance and the mean-squared error of the proposed estimator are computed. Asymptotic normality of the estimator is also established.

Keywords and Phrases: inhomogeneous Poisson process, intensity function, maximum likelihood estimator, consistency, bias, variance, mean-squared error, asymptotic normality.

1. Introduction

Let N be an inhomogeneous Poisson process on $[0,\infty)$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν and with (unknown) locally integrable intensity function λ, i.e., for any bounded Borel set B we have

$$\mu(B) = \mathbf{E}N(B) = \int_B \lambda(s)ds < \infty.$$

Furthermore, λ is assumed to be a power function, that is, for any $s \in [0,\infty)$, we can write $\lambda(s)$ as

$$\lambda(s) = as^b, \tag{1.1}$$

where a denotes (unknown) slope and b is a constant. It is assumed that we know b and $0 \le b < \infty$.

Here we consider a Poisson process on $[0,\infty)$ instead of, for instance, on $\mathbf{R}$ because λ has to satisfy (1.1) and must be non negative. For the same reason we also restrict our attention to the case $a > 0$.

Suppose now that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ (cf. (1.1)) is observed, though only within a bounded interval $W_n = [0, n] \subset [0, \infty)$. Our goal in this paper is to construct an estimator of λ at a given (fixed) point $s \in [0, n]$ using only a single realization $N(\omega)$ of the Poisson process N observed in interval $[0, n]$. We prove that the constructed estimator is consistent when the size of the window indefinitely expands. The asymptotic bias, variance and the mean-squared error of the proposed estimator are computed. We also establish the asymptotic normality of our estimator.

There are many practical situations where we have to use only a single realization for estimating intensity of a Poisson process. A review of such applications can be seen in [4], and a number of them can also be found in [2], [5], [7], [8] and [9].

Note that, if $b = 0$, then we have homogeneous Poisson process with rate $\lambda = a$. For this case, it is well-known that the maximum likelihood estimator of λ is given by

$$\hat{\lambda}_n = \frac{N([0, n])}{n}, \tag{1.2}$$

where $N([0, n])$ denotes the observed number of points in $[0, n]$. The present paper aims at extending this result to more general model given in (1.1). A related case also can be found in Example 2.7 of [7]. We refer to [6] for an excellent account of the theory of Poisson processes.

Note also that, the meaning of the asymptotic $n \to \infty$ in this paper is somewhat different from the classical one. Here n does not denote our sample size, but it denotes the length of the interval of observations. The size of our samples is a random variable denoted by $N([0, n])$.

2. Construction of the estimator and results

Let s_i, $i = 1, 2, \ldots, N([0, n])$, denote the locations of the points in the realization $N(\omega)$ of the Poisson process N (with intensity given by (1.1)), observed in interval $[0, n]$. Then, the likelihood function is given by

$$L = \exp\left\{-\frac{a}{b+1} n^{b+1}\right\} \prod_{i=1}^{N([0,n])} a s_i^b, \tag{2.1}$$

(cf. [1], p. 655).

The idea behind the construction of the likelihood function given in (2.1) can be described as follows. The likelihood function should be proportional to the following probability:

$$\begin{aligned} &\mathbf{P}(\text{there is exactly one point of realization in each} \\ &\{s_i\}, i = 1, 2, \ldots, N([0, n]), \text{ and no realization else where}). \end{aligned} \tag{2.2}$$

Since Poisson process has independent increment, the probability in (2.2) is equal to the multiplication of the two following probabilities:

$$\begin{aligned}&\mathbf{P}(\text{there is no realization in } [0,n] \setminus \cup_{i=1}^{N([0,n])}\{s_i\})\\&= \exp\left\{-\mu\left([0,n] \setminus \cup_{i=1}^{N([0,n])}\{s_i\}\right)\right\} \qquad (2.3)\end{aligned}$$

and

$$\begin{aligned}&\mathbf{P}(\text{there is exactly one point of realization in each set}\\&\{s_i\}, i = 1,2,\dots,N([0,n]))\\&= \prod_{i=1}^{N([0,n])} e^{-\mu(\{s_i\})}\mu(\{s_i\}). \qquad (2.4)\end{aligned}$$

Now note that, for each $i = 1,2,\dots,N([0,n])$, $\mu(\{s_i\})$ is proportional to $\lambda(s_i)$. Then, for our purpose, we may replace the r.h.s. of (2.4) by

$$\prod_{i=1}^{N([0,n])} e^{-\mu(\{s_i\})}\lambda(s_i). \qquad (2.5)$$

Multiplying the r.h.s. of (2.3) with the quantity in (2.5), yields

$$\begin{aligned}&\exp\left\{-\mu\left([0,n] \setminus \cup_{i=1}^{N([0,n])}\{s_i\}\right)\right\} \prod_{i=1}^{N([0,n])} e^{-\mu(\{s_i\})}\lambda(s_i)\\&= \exp\left\{-\mu\left([0,n] \setminus \cup_{i=1}^{N([0,n])}\{s_i\}\right)\right\}\\&\exp\left\{-\mu\left(\cup_{i=1}^{N([0,n])}\{s_i\}\right)\right\} \prod_{i=1}^{N([0,n])} \lambda(s_i)\\&= \exp\{-\mu([0,n])\} \prod_{i=1}^{N([0,n])} \lambda(s_i). \qquad (2.6)\end{aligned}$$

Clearly

$$\mu([0,n]) = \mathbf{E}N([0,n]) = \int_0^n as^b\, ds = \frac{an^{b+1}}{b+1}. \qquad (2.7)$$

By (2.7) and the fact $\lambda(s_i) = as_i^b$ (cf. (1.1)), we see that the r.h.s. of (2.6) is equal to the r.h.s. of (2.1).

Before defining an estimator of $\lambda(s)$, we first derived the maximum likelihood estimator $\hat{a}_n$ of a. To do this, note that

$$\begin{aligned}\ln L &= -\frac{a}{b+1}n^{b+1} + \sum_{i=1}^{N([0,n])} \ln(as_i^b)\\&= -\frac{a}{b+1}n^{b+1} + N([0,n])\ln a + \sum_{i=1}^{N([0,n])} \ln(s_i^b). \qquad (2.8)\end{aligned}$$

Maximizing $\ln L$ in (2.8) gives us:

$$\frac{\partial \ln L}{\partial a} = -\frac{n^{b+1}}{b+1} + \frac{N([0,n])}{a} = 0 \iff a = \frac{(b+1)N([0,n])}{n^{b+1}}$$

and

$$\frac{\partial^2 \ln L}{\partial a^2} = -\frac{N([0,n])}{a^2} < 0,$$

which directly yields the maximum likelihood estimator of a, which is given by

$$\hat{a}_n := \frac{(b+1)N([0,n])}{n^{b+1}}. \tag{2.9}$$

Now we may define our estimator of the intensity function λ at a given (fixed) point $s \in [0,n]$ (cf. (1.1)) as follows

$$\hat{\lambda}_n(s) := \hat{a}_n s^b. \tag{2.10}$$

Note that, in the case $b = 0$ (homogeneous Poisson process), the estimator in (2.10) reduces to the one given in (1.2).

Theorem 2.1. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\mathbf{E}\hat{\lambda}_n(s) = \lambda(s) \tag{2.11}$$

and

$$Var\left(\hat{\lambda}_n(s)\right) = \frac{(b+1)as^{2b}}{n^{b+1}}, \tag{2.12}$$

which converges to 0, as $n \to \infty$.

Hence, $\hat{\lambda}_n(s)$ is unbiased estimator of $\lambda(s)$. From Theorem 2.1 we also directly obtain the following results.

Corollary 2.2. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{2.13}$$

as $n \to \infty$. In other words, $\hat{\lambda}_n(s)$ is a consistent estimator of $\lambda(s)$. In addition, we also have

$$MSE\left(\hat{\lambda}_n(s)\right) = \frac{(b+1)as^{2b}}{n^{b+1}}, \tag{2.14}$$

which converges to 0, as $n \to \infty$.

In the next theorem, we establish the complete convergence of $\hat{\lambda}_n(s)$. We say $\hat{\lambda}_n(s)$ converges completely to $\lambda(s)$, as $n \to \infty$, if $\sum_{n=1}^{\infty} \mathbf{P}(|\hat{\lambda}_n(s) - \lambda(s)| > \epsilon) < \infty$, for every $\epsilon > 0$.

Theorem 2.3. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\hat{\lambda}_n(s) \xrightarrow{c} \lambda(s), \tag{2.15}$$

as $n \to \infty$.

Theorem 2.3 and the Borel-Cantelli lemma (e.g. see [3]) implies the following result.

Corollary 2.4. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\hat{\lambda}_n(s) \xrightarrow{a.s.} \lambda(s), \tag{2.16}$$

as $n \to \infty$. In other words, $\hat{\lambda}_n(s)$ is a strongly consistent estimator of $\lambda(s)$.

We conclude this section by establishing the asymptotic normality of $\hat{\lambda}_n(s)$, properly normalized, which is given in the following theorem.

Theorem 2.5. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$n^{(b+1)/2}\left(\hat{\lambda}_n(s) - \lambda(s)\right) \xrightarrow{d} Normal\left(0, a(b+1)s^{2b}\right), \tag{2.17}$$

as $n \to \infty$.

3. Proofs

Proof of Theorem 2.1

Using (2.9), (2.10) and (2.7), we directly obtain

$$\mathbf{E}\hat{\lambda}_n(s) = \frac{(b+1)s^b}{n^{b+1}}\mathbf{E}N([0,n]) = \frac{(b+1)s^b}{n^{b+1}}\left(\frac{an^{b+1}}{b+1}\right) = as^b = \lambda(s).$$

Similarly, the variance of $\hat{\lambda}_n(s)$ can easily be computed as follows

$$\begin{aligned} Var(\hat{\lambda}_n(s)) &= \left(\frac{(b+1)s^b}{n^{b+1}}\right)^2 Var(N([0,n])) = \left(\frac{(b+1)s^b}{n^{b+1}}\right)^2 \mathbf{E}N([0,n]) \\ &= \frac{(b+1)^2 s^{2b}}{(n^{b+1})^2}\left(\frac{an^{b+1}}{b+1}\right) = \frac{(b+1)as^{2b}}{n^{b+1}}. \end{aligned}$$

This completes the proof of Theorem 2.1.

In the proof of Theorem 2.3, we require the following lemma.

Lemma 3.1. *Let X be a Poisson r.v. with $\mathbf{E}X > 0$. Then, for any $\epsilon > 0$, we have*

$$\mathbf{P}\left(\frac{|X - \mathbf{E}X|}{\sqrt{\mathbf{E}X}} > \epsilon\right) \leq 2\exp\left\{-\frac{\epsilon^2}{2 + \epsilon/\sqrt{\mathbf{E}X}}\right\}.$$

Proof: We refer to [8], p. 222.

Proof of Theorem 2.3

To prove Theorem 2.3, we have to show that, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\hat{\lambda}_n(s) - \lambda(s)| > \epsilon\right) < \infty. \tag{3.1}$$

By (2.11), the probability appearing in (3.1) can be written as

$$\begin{aligned}
& \mathbf{P}\left(|\hat{\lambda}_n(s) - \mathbf{E}\hat{\lambda}_n(s)| > \epsilon\right) \\
= \; & \mathbf{P}\left(\frac{(b+1)s^b}{n^{b+1}}|N([0,n]) - \mathbf{E}N([0,n])| > \epsilon\right) \\
= \; & \mathbf{P}\left(\frac{|N([0,n]) - \mathbf{E}N([0,n])|}{\sqrt{\mathbf{E}N([0,n])}} > \frac{\epsilon\, n^{b+1}}{(b+1)s^b\sqrt{\mathbf{E}N([0,n])}}\right) \\
= \; & \mathbf{P}\left(\frac{|N([0,n]) - \mathbf{E}N([0,n])|}{\sqrt{\mathbf{E}N([0,n])}} > \frac{\epsilon\, n^{(b+1)/2}}{s^b\sqrt{a(b+1)}}\right).
\end{aligned} \tag{3.2}$$

Here we have used (2.7). By Lemma 3.1, the probability on the r.h.s. of (3.2) does not exceed

$$\begin{aligned}
& 2\exp\left\{-\frac{\epsilon^2 n^{b+1}}{a(b+1)s^{2b}\left(2 + \epsilon n^{(b+1)/2}s^{-b}(a(b+1)\mathbf{E}N([0,n]))^{-1/2}\right)}\right\} \\
& = 2\exp\left\{-\frac{\epsilon^2 n^{b+1}}{a(b+1)s^{2b}\left(2 + \epsilon(as^b)^{-1}\right)}\right\}.
\end{aligned} \tag{3.3}$$

Here we have used (2.7). Since the quantity on the r.h.s. of (3.3) is summable, then we obtain (2.15). Note that, if we only prove (2.15) for the case $b > 0$, then, instead of using Lemma 3.1, we can just simply use Chebyshev's inequality to obtain an upper bound for the probability on the l.h.s. of (3.2). This because the r.h.s. of (2.12) is summable for $b > 0$. This completes the proof of Theorem 2.3.

Proof of Theorem 2.5

By (2.9), (2.10) and (2.11), the l.h.s. of (2.17) can be written as

$$\begin{aligned}&\frac{(b+1)s^b}{n^{b+1}}\left(N([0,n])-\mathbf{E}N([0,n])\right)\\&=\frac{(b+1)s^b\sqrt{\mathbf{E}N([0,n])}}{n^{b+1}}\left(\frac{N([0,n])-\mathbf{E}N([0,n])}{\sqrt{\mathbf{E}N([0,n])}}\right).\end{aligned}\tag{3.4}$$

By (2.7), the r.h.s. of (3.4) reduces to

$$\sqrt{a(b+1)}\ s^b\left(\frac{N([0,n])-\mathbf{E}N([0,n])}{\sqrt{\mathbf{E}N([0,n])}}\right).\tag{3.5}$$

By the normal approximation for the Poisson distribution, the quantity in (3.5) can be written as

$$\sqrt{a(b+1)}\ s^b\left(Normal(0,1)+o_p(1)\right),$$

which converges in distribution to $Normal(0, a(b+1)s^{2b})$, as $n \to \infty$. This completes the proof of Theorem 2.5.

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